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On Minimal Interpolating Projections and Trace Duality

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We construct a two-dimensional subspace $V \subset C(K)$ such that an interpolating projection on V is a minimal projection with the norm >1. That answers a question posed by B. L. Chalmers. It also answers a question posed implicitly by a theorem of P. Morris and E. W. Cheney. We also give a quantitative generalization of the above mentioned theorem. As is suggested by the title, we use trace duality to obtain these results.

1. INTRODUCTION

This paper addresses a question posed by E. W. Cheney and K. H. Price:

PROBLEM 1.1 [3, Problem 15]. *For what subspaces V in $C(K)$ is it true that at least one of the minimal projections of $C(K)$ onto V is an interpolating projection?*

Here K is compact Hausdorff space and $C(K)$ is the space of all continuous real-valued functions on K.

DEFINITION 1.2. For a finite-dimensional subspace $V \subset C(K)$ we define the projectional constant of V to be

$$\lambda(V) = \inf\{\|P\| : P \text{ projection from } C(K) \text{ onto } V\}.$$

DEFINITION 1.3. We say that a projection P from $C(K)$ onto V is minimal if

$$\|P\| = \lambda(V).$$

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DEFINITION 1.4. A projection P from $C(K)$ onto V is called an interpolating projection if

$$Pf = \sum_{j=1}^{n} f(k_j) v_j,$$

where $(k_j) \subset K$; $(v_j) \subset V$ is a basis of V and $v_i(k_j) = \delta_{ij}$, $i, j = 1, ..., n$ and $n = \dim V$.

An easy consequence of a well-known theorem of Nachbin [6] is:

PROPOSITION 1.5 (cf. [4]). *Let V be an n-dimensional subspace of $C(K)$ with $\lambda(V) = 1$. Then there exists an interpolating projection P from $C(K)$ onto V such that P is a minimal projection.*

The following problem was open:

PROBLEM 1.6. *Does the converse to Proposition 1.5 hold?*

In Section 3 of this paper (Theorem 3.1) we construct an example of a subspace $V \subset C(K)$ such that an interpolating projection is a minimal projection onto V yet $\lambda(V) > 1$, hence giving a negative answer to Problem 1.6.

This example also provides a counterexample (cf. Proposition 3.5) to

CONJECTURE 1.7 (Chalmers [1]). *Let V be a finite-dimensional subspace of a Banach space X. Let P be a minimal projection from X onto V. Then P^* is a minimal projection from X^* onto the range of P^*.*

Theorem 4.2 answers Problem 1.1 in the finite-dimensional case.

DEFINITION 1.8. Let P be a projection from $C(K)$ onto V. Define $\Lambda_P: K \to R$ by

$$\Lambda_P(k) = \sup\{|(Pf)(k)| : \|f\| \leqslant 1\}.$$

Clearly $\sup\{|\Lambda_P(k)|, k \in K\} = \|P\|$.

In an attempt to solve Problem 1.1, Cheney and Morris proved the following:

THEOREM 1.9. (cf. [2]). *Let V be an n-dimensional Chebyshev subspace of $C(K)$ which admits a minimal interpolating projection P. Then either $\|P\| = 1$ or*

$$\# \{k: \Lambda_P(k) = \|P\|\} > n.$$

(Here # stands for the cardinality of the set).

In Section 4 (Theorem 4.8) we obtain the following generalization of this result:

Let P be a minimal interpolating projection from $C(K)$ onto an n-dimensional Chebyshev subspace V. Then

$$\#\{k: \Lambda_P(k) = \|P\|\} > n + (\|P\| - 1)^2.$$

A few words about the methods employed in this paper.

We mostly deal with a finite set K. Hence $C(K) = l_\infty^m$, where $m = \#K$. Let (e_j) be a canonical vector basis in $\mathbf{R}_m$. If dim $V = n$ then an interpolation projection from l_∞^m onto V is a projection of the form

$$Px = \sum_{j=1}^{n} \langle x, e_j \rangle v_j,$$

i.e., the point evaluation functionals in this case are vectors e_j, $j = 1, ..., n$, considered as the elements of the space l_1^n.

We use an idea that goes at least as far back as Cheney and Price [3] and Cheney and Morris [2] to describe a minimal projection as a solution to a best approximation problem.

Let $\mathscr{L}(l_\infty^m)$ be the space of linear bounded operators from l_∞^m into itself. Let $\mathscr{A} \subset \mathscr{L}(l_\infty^m)$ be a subspace of $\mathscr{L}$ defined by

$$\mathscr{A} := \{A \in \mathscr{L}(l_\infty^m) : \text{Range } A \subset V \subset \ker A\}.$$

Then P is a minimal projection from l_∞^m onto V *iff* 0 is the best approximation to P from $\mathscr{A}$. Just as in [2, 3] we use the dual characterization of best approximation to conclude that P is minimal if and only if there exists a functional φ in $[\mathscr{L}(l_\infty^m)]^*$ such that φ annihilates $\mathscr{A}$ (i.e., $\varphi(A) = 0$ for all $A \in \mathscr{A}$) and

$$|\varphi(P)| = \|P\| \, \|\varphi\|.$$

The relative "novelty" here is the use of trace duality to describe the functionals on $\mathscr{L}(l_\infty^m)$. While the trace duality is frequently used in Banach space theory (cf. [5, 7]) it is especially transparent for $\mathscr{L}(l_\infty^m)$. For convenience we reprove the needed results in Section 2.

In Section 3 we construct a counterexample to Problem 1.6 and Conjecture 1.

In Section 4 we derive a matrix equation which gives a necessary and sufficient condition for a space to have interpolating as a minimal one. As a corollary we derive the generalization of the Cheney–Morris Theorem 2.

2. TRACE DUALITY

In this section we introduce some theoretical aspects of trace duality for the spaces $\mathscr{L}(l_\infty^m)$. Although most of this theory is well-known for general Banach spaces, we find it convenient to prove the results for this particular case emphasizing the specific details needed in the next sections.

We use $e_1, ..., e_m$ to denote the canonical bases in $\mathbf{R}_m$, and $\langle \cdot, \cdot \rangle$ to denote the canonical inner product in $\mathbf{R}_m$. l_∞^m and l_1^m stand for $\mathbf{R}_m$ equipped with the norms

$$\|x\|_\infty = \max\{|\langle x, e_j \rangle|, j = 1, ..., m\},$$

$$\|x\|_1 = \sum_{j=1}^{m} |\langle x, e_j \rangle|$$

respectively.

For $y \in l_1^m$ and $v \in l_\infty^m$ we use $y \otimes v$ to define an operator in $\mathscr{L}(l_\infty^m)$ as

$$[y \otimes v](x) := \langle y, x \rangle v.$$

Hence if $A \in \mathscr{L}(l_\infty^m)$ is given by a matrix $A = (a_{ij})$ then

$$A = \sum u_j \otimes e_j,$$

where u_j are the rows vectors of the matrix A. For $A \in \mathscr{L}(l_\infty^m)$ given by a matrix (a_{ij}) we define

$$\|A\| := \max\left\{\sum_{j=1}^{m} |a_{ij}|, i = 1, ..., m\right\}.$$

DEFINITION 2.1. The nuclear norm of $A = (a_{ij})$ is defined by

$$v(A) := \sum_{j=1}^{m} \max_i |a_{ij}|.$$

Next we well need a somewhat unusual notation.

DEFINITION 2.2. Let $A = (a_{ij}) \in \mathscr{L}(l_\infty^m)$. We use $\tilde{\Sigma}(A)$ to denote the class of matrices (s_{ij}) defined by

$$s_{ij} = \begin{cases} 1 & \text{if } a_{ij} > 0 \\ -1 & \text{if } a_{ij} < 0 \\ \text{any number from } [-1, 1] \text{ if } a_{ij} = 0. \end{cases}$$

Definition 2.3. For a $t \times m$ matrix $A = (a_{ij})$ define an extremal set

$$\mathscr{T}(A) = \left\{ i\colon \sum_{j=1}^{m} |a_{ij}| = \|A\|,\ i = 1, ..., t \right\}.$$

Proposition 2.4. *Let $A = (a_{ij}) \in \mathscr{L}(l_\infty^m)$; $\Gamma = (\gamma_{ij}) \in \mathscr{L}(l_\infty^m)$. Then*

1. $|\mathrm{tr}(A\Gamma)| \leqslant \|A\|\ \nu(\Gamma)$,

2. *The equality in 1 is attained if and only if $\Gamma = \pm[\Sigma(A)]^T D$ where $[\Sigma(A)] \in \tilde{\Sigma}(A)$ and $D = [d_1, ..., d_m]$ is a diagonal matrix*

$$D := \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_m \end{bmatrix}$$

with $d_j \geqslant 0$ and $d_j = 0$ if $j \notin \mathscr{T}(A)$,

3. *In particular for any matrix A there exists Γ such that* $\mathrm{tr}(A\Gamma) = \|A\|\ \nu(\Gamma)$.

Proof. To prove 1, observe that

$$|\mathrm{tr}\ A\Gamma| = \left| \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}\gamma_{ji} \right| \leqslant \sum_{i=1}^{m} (\max_j |\gamma_{ji}|) \left(\sum_{j=1}^{m} |a_{ij}| \right)$$
$$\leqslant \max_i \left(\sum_{j=1}^{m} |a_{ij}| \right) \cdot \sum_{i=1}^{m} \max_j |\gamma_{ji}|. \tag{2.1}$$

To prove 2, note that the first inequality in (2.1) is an equality if and only if each column of the matrix Γ is a positive constant multiple of the vector (sign a_{ij}), $j = 1, ..., m$, while the second inequality is an equality if and only $\gamma_{ij} = 0$ if $j \notin \mathscr{T}(A)$.

Finally 3 follows from 2. ∎

Proposition 2.2. *Let $N(l_\infty^m)$ stand for the space of all $m \times m$ matrices equipped with the nuclear norm ν. Then $\mathscr{N}(l_\infty^m)$ is isometrically isomorphic to $[\mathscr{L}(l_\infty^m)]^*$.*

Moreover, every functional $\varphi \in [\mathscr{L}(l_\infty^{(m)})]^*$ is uniquely identified with a $B \in \mathscr{N}(l_\infty^m)$ by

$$\varphi_B(A) = \mathrm{tr}(AB), \qquad \textit{for all} \quad A \in \mathscr{L}(l_\infty^m),$$
$$\|\varphi_B\| = \nu(B).$$

Proof. Since $m^2 = \dim \mathscr{L}(l_\infty^m) = \dim[\mathscr{L}(l_\infty^m)]^* = \dim \mathscr{N}(l_\infty^m)$ we conclude that the spaces in question are algebraically isomorphic. Proposition 2.4(3) shows that the map $B \to \varphi_B$ is an isometric isomorphism. ∎

Let V_n be an n-dimensional subspace of l_∞^m.
Let $\mathscr{A}(V_n) := \{A \in \mathscr{L}(l_\infty^m) : \text{Range } A \subset V_n \subset \ker A\}$.
Let $\mathscr{P}(V_n) := \{P \in \mathscr{L}(l_\infty^m) : P \text{ is a projection from } l_\infty^m \text{ onto } V_n\}$.

PROPOSITION 2.6. *Let $P \in \mathscr{P}(V_n)$. Then P is minimal if and only if there exist $[\Sigma(P)] \in \tilde{\Sigma}(P)$ and a diagonal operator $D = [d_1, ..., d_m]$ such that*

$$\operatorname{tr}([\Sigma(P)]^T DA) = 0, \qquad \text{for all} \quad A \in \mathscr{A}(V_n). \tag{2.2}$$

Proof. Note that $P + \mathscr{A}(V_n) = \mathscr{P}(V_n)$, for every $P \in \mathscr{P}(V_n)$. Hence, P is minimal if and only if zero is the best approximation to P from $\mathscr{A}(V_n)$. Therefore P is minimal if and only if zero is the best approximation to P from $\mathscr{A}(V_n)$. Therefore P is minimal if and only if there exists a functional $\varphi \in [\mathscr{L}(l_\infty^m)]^*$ such that $\varphi(P) = \|P\| \|\varphi\|$ and $\varphi(A) = 0$ for all $A \in \mathscr{A}$.

According to Proposition 2.5 this is equivalent to the existence of an operator $\Gamma \in \mathscr{N}(l_\infty^m)$ with

(a) $\operatorname{tr}(P\Gamma) = \|P\| \, \nu(\Gamma)$,

(b) $\operatorname{tr}(A\Gamma) = 0$ for all $A \in \mathscr{A}(V_n)$.

Using Proposition 2.4(2) we conclude that Γ is of the form $[\Sigma(P)]^T D$. ∎

Remark 2.7. The space $V_n^\perp$ consists of all vectors $x \in l_1^m$ that annihilate V_n, i.e.,

$$V_n^\perp = \{x \in l_1^m : \langle x, v \rangle = 0 \text{ for all } v \in V_n\}.$$

Hence $\dim V_n^\perp = m - n$. Let $v_1, ..., v_n$ be a basis for V_n while $f_1, ..., f_{m-n}$ is a basis for $V_n^\perp$.

Then

$$\mathscr{A}(V_n) = \operatorname{span}\{f_i \otimes v_j;\, i = 1, ..., m-n;\, j = 1, ..., n\}.$$

Therefore $[\Sigma(A)]^T D$ annihilates $\mathscr{A}(V_n)$ if and only if

$$\operatorname{tr}([\Sigma(A)]^T D[f_i \otimes v_j]) = \langle f_i, [\Sigma(A)]^T D v_j \rangle = 0,$$

for all $i = 1, ..., m-n$; $j = 1, ..., n$.

Remark 2.8. Proposition 2.6 is equivalent to: P is minimal if and only if there exists a $\Gamma \in \mathscr{N}(l_\infty^m)$ with

(a) $\operatorname{tr}(P\Gamma) = \|P\| \, \nu(\Gamma)$,

(b) $\operatorname{tr}(A\Gamma) = 0$ for all $A \in \mathscr{A}(V_n)$.

Remark 2.9. A proposition similar to Proposition 2.5 can be proved for operators in $\mathscr{L}(l_1^m)$.

In this case the norm of $A = (a_{ij}) \in \mathscr{L}(l_1^m)$ is defined to be

$$\|A\| = \max\left\{\sum_{i=1}^{m} |a_{ij}|, j = 1, ..., m\right\},$$

while the norm in $\mathscr{N}(l_1^m)$ is

$$v(A) = \sum_{i=1}^{m} \max_j |a_{ij}|$$

and $\operatorname{tr}(AB) \leqslant \|A\|\, v(B)$.

3. The Main Example

THEOREM 3.1. *There exists a two-dimensional subspace $V_2 \subset l_\infty^6$ such that the interpolating projection $P = e_1 \otimes v_1 + e_2 \otimes v_2$ is minimal and* $\|P\| > 1$.

Proof. Let

$$v_1 = \left(1, 0, \frac{2+\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right),$$

$$v_2 = \left(0, 1, \frac{\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, -\frac{2+\sqrt{2}}{4}\right),$$

$$V_2 = \operatorname{span}\{v_1, v_2\} \subset l_\infty^6.$$

Consider the projection $P = e_1 \otimes v_1 + e_2 \otimes v_2$. In matrix form

$$P = \begin{bmatrix} 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ \frac{2+\sqrt{2}}{4}, & \frac{\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \\ \frac{\sqrt{2}}{4}, & \frac{2+\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \\ \frac{2+\sqrt{2}}{4}, & -\frac{\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \\ \frac{\sqrt{2}}{4}, & -\frac{2+\sqrt{2}}{4}, & 0, & 0, & 0, & 0 \end{bmatrix}.$$

Next we describe the space $\mathscr{A}(V_2)$. Note that the vector $f=(x_1, x_2, t_1, t_2, t_3, t_4)$ belongs to $V_2^\perp$ if and only if

$$0=\langle f, v_1\rangle = x_1+\frac{2+\sqrt{2}}{4}t_1+\frac{\sqrt{2}}{4}t_2+\frac{2+\sqrt{2}}{4}t_3+\frac{\sqrt{2}}{4}t_4,$$

$$0=\langle f, v_2\rangle = x_2+\frac{\sqrt{2}}{4}t_1+\frac{2+\sqrt{2}}{4}t_2-\frac{\sqrt{2}}{4}t_3-\frac{2+\sqrt{2}}{4}t_4.$$

Choosing (t_1, t_2, t_3, t_4) to be consecutively $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$ we obtain the basis f_1, f_2, f_3, f_4 for $V_2^\perp$ as

$$f_1=\left(-\frac{2+\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}, 1, 0, 0, 0\right),$$

$$f_2=\left(-\frac{\sqrt{2}}{4}, -\frac{2+\sqrt{2}}{4}, 0, 1, 0, 0\right),$$

$$f_3=\left(-\frac{2+\sqrt{2}}{4}, +\frac{\sqrt{2}}{4}, 0, 0, 1, 0\right),$$

$$f_4=\left(-\frac{\sqrt{2}}{4}, +\frac{2+\sqrt{2}}{4}, 0, 0, 0, 1\right).$$

Now $\mathscr{A}(V_2)$ can be written as a linear span of $A_i^{(j)}$ with

$$A_i^{(j)}=f_j\otimes v_i; \qquad j=1, ..., 4; i=1, 2.$$

Next pick the operator Γ to be

$$\Gamma=\begin{bmatrix} 0, & 0, & 1, & 1, & 1, & 1 \\ 0, & 0, & 1, & 1, & -1, & -1 \\ 0, & 0, & 1, & 1, & 1, & 0 \\ 0, & 0, & 1, & 1, & 0, & -1 \\ 0, & 0, & 1, & 0, & 1, & 1 \\ 0, & 0, & 0, & -1, & 1, & 1 \end{bmatrix}.$$

By Proposition 2.4(2) (or by direct calculation) we have

$$\operatorname{tr}(P\Gamma)=\|P\|\,\nu(\Gamma)=4\times\frac{2+2\sqrt{2}}{4}=2(1+\sqrt{2}),$$

$$\|P\|=\frac{2+2\sqrt{2}}{4}=\frac{1+\sqrt{2}}{2}>1.$$

To prove that P is minimal (cf. Remark 2.8) it suffices to prove that

$$0=\operatorname{tr}(\Gamma A_i^{(j)})=f_j(\Gamma v_i); \qquad i=1, 2; j=1, 2, 3, 4.$$

We have

$$\Gamma v_1 = \left(1+\sqrt{2}, 0, 1+\frac{3\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}, 1+\frac{3\sqrt{2}}{4}, \frac{2+\sqrt{2}}{4}\right),$$

$$\Gamma v_2 = \left(0, 1+\sqrt{2}, \frac{2+\sqrt{2}}{4}, \frac{4+3\sqrt{2}}{4}, -\frac{2+\sqrt{2}}{4}, -\frac{4+3\sqrt{2}}{4}\right).$$

It is now easy to verify that $0 = f_j(\Gamma v_1) = f_j(\Gamma v_2)$; $j = 1, 2, 3, 4$. ∎

Remark 3.2. The space V_2 in the previous theorem is in fact a symmetric Banach space in the sense that

$$\|\alpha v_1 + \beta v_2\| = \|\beta v_1 + \alpha v_2\| = \|\alpha v_1 - \beta v_2\| \qquad \text{for all} \quad \alpha, \beta \in \mathbf{R}.$$

Remark 3.3. The space V_2 constructed above is a Chebyshev subspace of $C(K)$, where $K = \{1, 2, 3, 4, 5, 6\}$ in the sense that V_2 restricted to any two points is two-dimensional. Equivalently every 2×2 submatrix of the matrix

$$\begin{bmatrix} 1 & 0 & \frac{2+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 1 & \frac{\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & -\frac{2+\sqrt{2}}{4} \end{bmatrix}$$

is invertible.

Remark 3.4. It is possible to show that "six" (the dimension of l_∞^6) is the least possible. If V_n is a subspace of l_∞^m with $m \leqslant 5$ that admits a minimal interpolating projection then $\lambda(V_n) = 1$.

PROPOSITION 3.5. *The projection P constructed above provides a counter-example to Conjecture* 1, *i.e., P^* is not a minimal projection from l_1^m onto the range of P^*.*

Proof. Clearly the range of P^* is $\text{span}\{e_1, e_2\} \subset l_1^6$. Consider the projection Q from l_1^6 onto $\text{span}\{e_1, e_2\}$ defined as

$$Qx = \langle x, e_1 \rangle e_1 + \langle x, e_2 \rangle e_2.$$

Then $\|Qx\|_1 = |\langle x, e_1 \rangle| + \langle x, e_2 \rangle| \leqslant \sum_{j=1}^{6} \langle x, e_j \rangle = \|x\|_1$. Hence $1 = \|Q\| < \|P^*\|$ since $\|P^*\| = \|P\| > 1$. ∎

4. CHENEY–MORRIS THEOREM

In this section we will assume that V_n is an n-dimensional subspace of l_∞^{n+k}. Every interpolating projection from l_∞^{n+k} onto V_n can be written as

$P=\sum_{j=1}^{n} e_{k_j}\otimes v_j$, where $(k_1, ..., k_n)$ is a collection of distinct integers between 1 and $n+k$. In this case it will be convenient for us to consider a permutation of the natural basis of l_∞^{n+k} and of its dual so that the same projection P can be written as a block matrix

$$P=\begin{bmatrix} I_{n\times 2} & 0 \\ B^T & 0 \end{bmatrix},$$

where B is a given $n\times k$ matrix (b_{ij}).

Hence we will treat (e_{k_j}) as the first n coordinates and write the projection as $\sum e_j\otimes v_j$.

Correspondingly we will think of l_∞^{n+k} as $l_\infty^n\oplus l_\infty^k$ and use the notation $[x, y]$ to denote a vector in l_∞^{n+k}, where $x\in l_\infty^n$ consists of the first n coordinates in the permuted basis ($(k_1, ..., k_n)$ coordinates of the standard basis) and $y\in l_\infty^k$ represent the remaining k coordinates. The space V_n is thus spanned by vectors $v_j=[e_j, b_j]$, where $b_j=(b_{j1}, ..., b_{jk})\in l_\infty^k$ are the same as the rows of the matrix B appearing above.

Remark 4.1. If $\|P\|=1$ then $1, 2, ..., n\in\mathscr{T}(P)$ and hence

$$\#\mathscr{T}(P)\geqslant n.$$

THEOREM 4.2. *Let $\|P\|>1$. Then P is minimal if and only if there exists a $\Sigma(B)\in\tilde{\Sigma}(B)$ that satisfies the equation*

$$B\bar{D}[\Sigma(B)]^T B=B\bar{D}\Lambda, \tag{4.1}$$

where $\bar{D}=[d_1, ..., d_k]$ is a $k\times k$ non-zero diagonal matrix with non-negative entries and with $d_j=0$ if $j\notin\mathscr{T}(B^T)$ (see Definition 2.3) and $\Lambda=(\lambda_{ij})$ is a $k\times k$ matrix with $|\lambda_{ij}|\leqslant 1$.

Remark 4.3. To illustrate Theorem 4.2 we write the appearing matrices explicitly in the case $n=2, j=3$:

$$v_1=(1, 0, b_{11}, b_{12}, b_{13}), \qquad v_2=(0, 1, b_{21}, b_{22}, b_{23}),$$

$$B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{22} & b_2 & b_{23} \end{bmatrix},$$

$$P=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ b_{11} & b_{21} & 0 & 0 & 0 \\ b_{12} & b_{22} & 0 & 0 & 0 \\ b_{13} & b_{23} & 0 & 0 & 0 \end{bmatrix}.$$

Assuming that

$$|b_{11}| + |b_{21}| = |b_{12}| + |b_{22}| = |b_{13}| + |b_{23}| = \|P\| > 1, \tag{4.2}$$

i.e., $\mathscr{T}(P) = \{3, 4, 5\}$,

$$\Gamma = \begin{bmatrix} 0 & 0 & d_1(\operatorname{sign} b_{11}) & d_2(\operatorname{sign} b_{12}) & d_3(\operatorname{sign} b_{13}) \\ 0 & 0 & d_1(\operatorname{sign} b_{21}) & d_2(\operatorname{sign} b_{22}) & d_3(\operatorname{sign} b_{23}) \\ 0 & 0 & d_1 \lambda_{11} & d_2 \lambda_{21} & d_3 \lambda_{31} \\ 0 & 0 & d_1 \lambda_{12} & d_2 \lambda_{22} & d_3 \lambda_{32} \\ 0 & 0 & d_1 \lambda_{13} & d_2 \lambda_{23} & d_3 \lambda_{33} \end{bmatrix}.$$

Then Theorem 4.2 reads: there exist $|\lambda_{ij}| \leqslant 1$, and $d_1 \geqslant 0$, $d_2 \geqslant 0$, $d_3 \geqslant 0$, $|d_1| + |d_2| + |d_3| > 0$, such that

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \operatorname{sign} b_{11} & \operatorname{sign} b_{21} \\ \operatorname{sign} b_{12} & \operatorname{sign} b_{22} \\ \operatorname{sign} b_{13} & \operatorname{sign} b_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix}. \tag{4.3}$$

Proof of Theorem 4.2. Assume P is minimal. Since $\|P\| > 1$, the norm $\|P\|$ is not attained in the first n rows of the matrix P. By Proposition 2.6 we conclude that there exists Γ of the form

$$\Gamma = [\Sigma(B)]^T D$$

that annihilates $\mathscr{A}(V_n)$, where

$$D = \begin{bmatrix} 0 & 0 \\ 0 & \bar{D} \end{bmatrix}.$$

From the form of P we have

$$\Gamma = \begin{bmatrix} 0 & [\Sigma(B)] \bar{D} \\ 0 & A^T \bar{D} \end{bmatrix}. \tag{4.4}$$

Let $f = [x, u] \in V_n^\perp$. Then

$$0 = \langle f, v_j \rangle = \langle x, e_j \rangle + \langle u, b_j \rangle, \qquad \text{for all} \quad j = 1, ..., n.$$

Observe that $b_j = B^T e_j$. Thus $\langle x, e_j \rangle + \langle u, B^T e_j \rangle = 0$ or

$$x = -Bu.$$

Hence $V_n^\perp = \{[-Bu, u] : u \in l_1^k\}$. The condition

$$\operatorname{tr}(\Gamma \cdot [f \otimes v_j]) = 0, \qquad \text{for all} \quad j = 1, ..., n \text{ and for all } f \in V_n^\perp$$

can be written as

$$f(\Gamma v_j) = 0, \qquad \text{for all} \quad j = 1, ..., n \text{ and for all } f \in V_n^\perp. \tag{4.5}$$

From the form of Γ and v_j we have

$$\Gamma v_j = [(\Sigma(B))\, \bar{D}b_j, A^T \bar{D}b_j] = [(\Sigma(B))\, \bar{D}B^T e_j, A^T \bar{D}B^T e_j].$$

Now the condition (4.5) reads

$$\langle Bu, [\Sigma(B)]\, \bar{D}B^T e_j \rangle = \langle u, A^T \bar{D}B^T e_j \rangle$$

or

$$\langle u, B^T [\Sigma(B)]\, \bar{D}B^T e_j \rangle = \langle u, A^T \bar{D}B^T e_j \rangle.$$

Since this equality holds for every $u \in \mathbf{R}_k$ and for all $j = 1, ..., n$ we have

$$B^T [\Sigma(B)]\, \bar{D}B^T = A^T \bar{D}B^T.$$

Transposing this equation we have

$$B\, \bar{D}[\Sigma(B)]^T B = B\, \bar{D}A.$$

Since every step above can be reversed, Eq. (4.1) implies that the matrix (4.4) satisfies the conditions of Remark 2.8 and hence the converse is also true. ∎

COROLLARY 4.4. *Assume the conditions of Theorem 4.2. Let B_0 be a submatrix of the matrix B consisting of the columns $i_j \in \mathcal{T}(P)$. Then P is minimal if and only if there exists a matrix $\Sigma(B_0) \in \tilde{\Sigma}(B_0)$ such that*

$$B_0\, \bar{D}_0 [\Sigma(B_0)]^T B_0 = B_0\, \bar{D}_0 A_0,$$

where $\bar{D}_0$ and A_0 are the appropriate submatrices of the matrices $\bar{D}$ and A.

Proof. Let $D^{1/2} = [\sqrt{d_1}, ..., \sqrt{d_n}]$. Then $\bar{D} = D^{1/2} \cdot D^{1/2}$.
From (4.1) we have

$$(BD^{1/2})(D^{1/2}[\Sigma(B)]^T) B = B\, \bar{D}A.$$

Since $d_j = 0$ if $j \notin \mathcal{T}(B^T)$ the non-zero columns of $BD^{1/2}$ coincide with $B_0 \bar{D}_0^{1/2}$ while non-zero rows of $D^{1/2}(\Sigma(B))^T$ coincide with the rows of $D_0^{1/2}(\Sigma(B_0))^T$ [cf. (4.3)]. Hence the non-zero columns and rows of the lefthand side of (4.1) coincide with $B_0 \bar{D}_0 [\Sigma(B_0)]^T B_0$. Similarly, the same holds for the right-hand side.

DEFINITION 4.5. We say that the space $V_n \subset l_\infty^{n+k}$ is Chebyshev if the $\dim(V_n \mid i_1, ..., i_n) = n$ for any integers $i_1 < i_2 < \cdots < i_n \leqslant n+k$.

In other words, every $n \times n$ submatrix of the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & b_{11} & \cdots & b_{1k} \\ 0 & 1 & \cdots & 0 & b_{21} & \cdots & b_{2k} \\ & & & & & & \\ 0 & 0 & \cdots & 1 & b_{n1} & \cdots & b_{nk} \end{bmatrix}$$

is invertible.

Next we will need two simple lemmas.

LEMMA 4.6. *Let E be a subspace at l_1^m. Then there exists a projection Q from l_1^m into l_1^m such that* $\ker Q = E$ *and* $\|Q\| < \sqrt{\dim E} + 1$.

Proof. By [5] there exists a projection R from l_1^m onto E with $\|R\| < \sqrt{\dim E}$.

Let $Q = I - R$. Then $\ker Q = \text{range } R = E$ and $\|Q\| \leqslant 1 + \|R\| < \sqrt{\dim E} + 1$.

LEMMA 4.7 (cf. [8]). *Let T be a right invertable operator from l_1^m onto l_1^q. Then the set of all projection $Q \in \mathscr{L}(l_1^m)$ such that*

$$\ker Q = \ker T$$

coincides with the set

$$\{ST : TS = I\}.$$

Proof. Trivially $TS = I$ implies that ST is a projection in $\mathscr{L}(l_1^m)$. Since S is left invertible we have $\ker S = \{0\}$ and

$$\ker ST = \ker T.$$

Conversely let $Q \in \mathscr{L}(l_1^m)$ be a projection with $\ker Q = \ker A$. Let $V_0 = \text{range } Q$.

Then $\dim V_0 = q$ and $l_\infty^m = \ker Q \otimes V_0$. Define an operator T_0 by restricting T to V_0:

$$T_0 = = T \mid V_0 : V_0 \to l_1^q.$$

Then $\ker T_0 = \{0\}$; T_0 maps a q-dimensional space into q-dimensional space. Hence T_0 is invertable. Clearly $S = T_0^{-1}Q$ is the right inverse to T and

$$Q = ST. \quad \blacksquare$$

We are now ready to prove the generalization of the Cheney–Morris Theorem:

THEOREM 4.8. *Let V_n be a Chebyshev subspace of l_∞^{n+k}. Let P be an interpolating projection from l_∞^{n+k} onto V_n such that $\|P\| = \lambda(V_n) > 1$. Then*

$$\# \mathcal{T}(P) > n + (\|P\| - 1)^2.$$

Proof. In view of Corollary 4.4 we may assume that $\# \mathcal{T}(P) = k$. Consider two cases. First, let $k \leqslant n$. Then the matrix B from Theorem 4.2 is left invertible. Hence (from (4.1))

$$\bar{D}[\Sigma(B)]^T B = \bar{D}A.$$

Suppose that $d_j \neq 0$. Then the j-s diagonal entry of $\bar{D}[\Sigma(B)]^T B$ is $d_j \sum_{i=1}^n |b_{ij}| = d_j \|P\|$ while the j-s diagonal entry of $\bar{D}A$ is $d_j \lambda_{ij}$. Hence $d_j \|P\| = d_j \lambda_{ij} = d_j |\lambda_{ij}| \leqslant d_j$. So $\|P\| \leqslant 1$ which contradicts the assumption of the theorem.

Now suppose $k > n$. Then dim ker $B = k - n$ and B is invertible from the right. By Lemmas 4.6 and 4.7 we can choose a right inverse S such that $\|SB\| < \sqrt{k-n} + 1$. Here $SB: l_1^k \to l_1^k$. From (4.1) we have

$$\operatorname{tr}(B \bar{D}[\Sigma(B)]^T) = \operatorname{tr}(B \bar{D}AS) = \operatorname{tr}(SB \bar{D}A).$$

By direct computation (cf. (4.3))

$$\operatorname{tr}(B \bar{D}[\Sigma(B)]^T = (\Sigma d_j) \|P\|.$$

By trace-duality for $\mathcal{L}(l_1^k)$ (cf. Remark 2.9)

$$\operatorname{tr}(SB \bar{D}A) \leqslant \|SB\| \, \nu(\bar{D}A) \leqslant (\sqrt{k-n} + 1)(\Sigma d_j).$$

Hence

$$\|P\| < (\sqrt{k-n} + 1),$$

which implies

$$k - n > (\|P\| - 1)^2$$

or

$$\# \mathcal{T}(P) = k > n + (\|P\| - 1)^2. \quad \blacksquare$$

Remark 4.9. The proof of Theorem 4.8 strongly depends on the fact that V_n is Chebyshev. Otherwise, we could not conclude the invertibility of the matrix B.

Without this assumption we can show that if $V_n \subset l_\infty^{n+2}$ admits a minimal interpolating projection then $\lambda(V_n) = 1$.

As we mentioned earlier for $n = 2$ we can improve this to $V_2 \subset l_\infty^5$.

Remark 4.10. From (4.1) it is easy to deduce that if P is minimal and the vector v_j has the property $v_{ji} v_{jk} \geqslant 0$ for all j, i, k then $\|P\| = 1$.

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